Direct Statistical Analysis of Nonlinear Systems: CADET

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The Covariance Analysis Describing Function Technique—CADET—is a new and powerful computerized approach to the direct statistical analysis of nonlinear systems through the combination of linear system covariance analysis and multiple-input describing functions. CADET is an excellent system design and analysis tool which provides for efficient computation of the mean and covariance of high-order, multiple nonlinearity systems with multiple statistically described inputs. Although the results are approximate, the computational cost can be considerably below that of Monte Carlo methods. The technique is briefly developed herein and applied to systems arising in the context of guidance and control.

Introduction

IN order to perform meaningful analytic studies, it is necessary to employ a realistic mathematical description of the system under consideration. Concise well-developed mathematical techniques, such as covariance analysis, are available for the statistical analysis of linear systems. However, one often encounters systems in which significant—i.e., non-negligible—nonlinearities exist. These problems may include filters, linear or otherwise, or may simply involve statistical signal propagation. In either case, the existence of significant nonlinear behavior has traditionally necessitated the employment of Monte Carlo techniques—repeated simulation trials plus ensemble averaging—to arrive at a statistical description of system behavior. Hundreds or even thousands of sample responses are often needed to obtain statistically meaningful results, and the corresponding computer burden can be exceptionally severe both in cost and time. This is particularly true of design sensitivity and parameter tradeoff studies. Thus, one is led to search for an alternative method of analysis which is computationally efficient and can provide for the realistic assessment of nonlinear system performance.

An exceptionally promising technique, developed specifically for the direct statistical analysis of dynamic nonlinear systems, is presented herein. It is called covariance analysis describing function technique (CADET). This technique employs the device of statistical linearization of nonlinear system elements in conjunction with ordinary covariance analysis techniques to yield statistical performance projections. The technique of statistical linearization, per se, is well-known and thoroughly documented elsewhere.[1] The joint use of statistical linearization and linear system covariance analysis has been applied to the problem of nonlinear filtering.[2] Other somewhat related investigations[3,4] also treat the subject of nonlinear filtering. The present paper deals not with nonlinear filtering but rather with the statistical analysis of given nonlinear systems. This is a fundamentally different focus from the other work cited.

The basic CADET theoretical development along with illustrative examples of its application are presented herein. It should be noted at the outset that CADET results are approximate. Of course, so are the computed statistics of a finite number of Monte Carlo samples an approximation to the true statistics. Computational efficiency and over-all simplicity of CADET are felt to far outweigh the fact that the technique is not exact.

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Methodology

The general form of the system to be considered is illustrated in Fig. 1, where x is the system state vector,‡ w is a white noise input vector, and $f(x)$ is a vector nonlinear function. The objective is to determine the statistical properties—mean and second-moment—of $x(t)$. The success of CADET in achieving this goal depends on how well $f(x)$ can be approximated.

The most common technique for dealing with nonlinear systems is to construct a linear model to approximate $f(x)$. A small-signal linearization can be obtained by expanding $f(x)$ in a Taylor series about some operating point—i.e., nominal state vector—and retaining the linear terms of the expansion. This approximates $f(x)$ for small perturbations about the operating point. Inputs which exceed the limits of reasonable linear approximation are handled by repeatedly relinearizing about new operating points. However, there are many important nonlinear relationships which exhibit discontinuities—e.g., quantizers, two-level switches, etc.—for which linearization in the ordinary sense is not possible. This is a major restriction which limits the applicability of so-called true linearization. Another more subtle restriction is the fact that the system state vector, per se, is involved in the linearization procedure. Hence it must be known and, as a consequence, any results obtained are specific to the particular trial under consideration. This viewpoint thus leads to Monte Carlo approaches.

The approximation of a nonlinear operation by a linear one which depends on some properties of the input signal is often referred to as quasi-linearization.[1] This procedure results in different linear approximations for the same nonlinearity with different input signal forms. Unlike linear approximators, quasi-linear approximators can accommodate any range of signal magnitudes and are in some sense dependent on input signal amplitude—a basic characteristic of nonlinear systems. Quasi-linearization and true linearization of a hard saturation

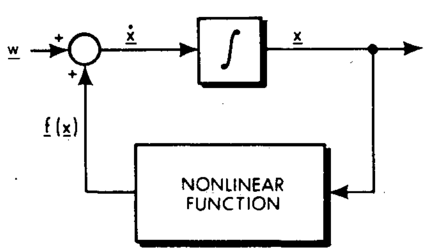

Fig. 1 System block diagram.

‡ Vectors are denoted with an underscore in all figures.

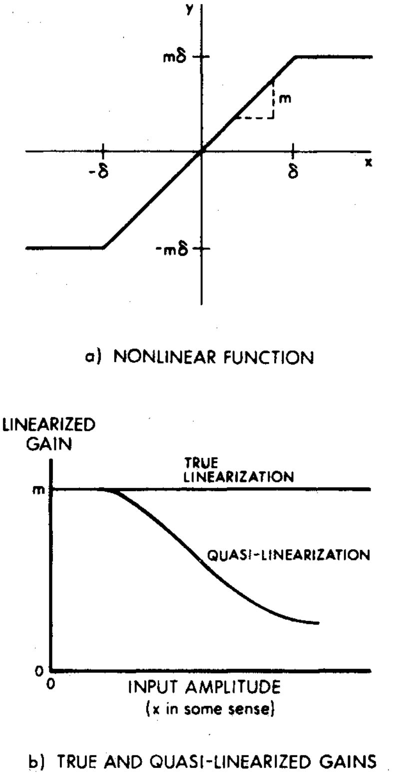

Fig. 2 Illustration of quasi-linearization.[1]

nonlinearity (limiter) are illustrated in Fig. 2. True linearization at the origin results in a constant gain for all input signals, whereas quasi-linearization produces a gain which decreases for larger input signal amplitudes; this is intuitively satisfying.

This input signal dependency requires that quasi-linearization be done for specified input signal forms. A practical solution of this problem for feedback system configurations depends on avoiding the calculation of the input signal form by assuming it to have a form which is guessed in advance. This leads us to consider certain basic signal forms with which to derive quasi-linear approximators for nonlinear operators. Three of the most fundamental signal forms are the bias, sinusoid, and gaussian random process. These forms of the assumed input signal are taken to be the principal basis for the calculation of quasi-linear approximators, and linear combinations of these and other signal forms can be used to calculate multiple-input approximators. Quasi-linear approximators of this type, which approximate the transfer characteristics of the nonlinearity, are termed describing functions and have received extensive treatment by Gelb and Vander Velde.[1] The major limitation on the

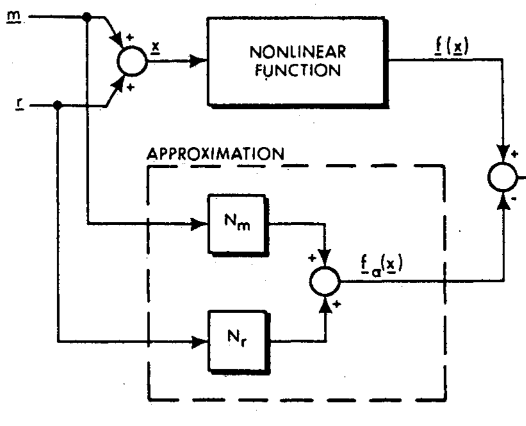

Fig. 3 Quasi-linear approximation.

use of these describing functions to characterize nonlinear system response lies in the requirement that the actual input signal to the nonlinearity approximate the form of the signal used to derive the describing functions. Experience with a large class of practical systems has shown this not to be as restrictive as it would first appear.

The assumed input signal forms for statistical linearization shall be restricted to the bias and gaussian random processes in the following development of CADET. Statistical linearization herein simply refers to the quasi-linearization of a given nonlinearity subject to these input forms.

Minimum Variance Quasi-Linear Approximators

Consider approximation of the nonlinear function $f(x)$ in Fig. 1 by a linear function, in the sense suggested by Fig. 3. The input to the nonlinear function, x, is taken to be comprised of a mean, m, plus a zero-mean, independent, random-process, r. Thus,

$$x = m + r \tag{1}$$

The mean can arise because of an average value of w, a mean initial value of x, a rectification in the nonlinearity, or a combination of these. The approximating output, $f_a(x)$, is comprised of the sum of two terms, one linearly related to m and the other linearly related to r. The so-called multiple-input describing function gain matrices, N_m and N_r, are chosen to minimize the mean square error in representation of the actual nonlinearity output $f(x)$ by $f_a(x)$.

Calculation of N_m and N_r is readily accomplished. Note first from Fig. 3 that

$$\begin{aligned} e &= f(x) - f_a(x) \\ &= f(x) - N_m m - N_r r \end{aligned} \tag{2}$$

Forming the matrix ee^T, the mean square approximation error is minimized by computing

$$(\partial/\partial N_m)(\text{trace } E[ee^T]) = (\partial/\partial N_r)(\text{trace } E[ee^T]) = 0 \tag{3}$$

where $E[\ ]$ denotes ensemble expectation. These computations result in the relationships

$$N_m mm^T = E[f(x)]m^T \tag{4}$$

and

$$N_r E[rr^T] = E[f(x)r^T] \tag{5}$$

since $E[rm^T] = E[mr^T] = 0$. Equations (4) and (5) define N_m and N_r. Denoting the random-process covariance matrix by $S(t)$, viz.,

$$S(t) = E[r(t)r^T(t)] \tag{6}$$

it follows from Eq. (5) that

$$N_r(m, S) = E[f(x)r^T]S^{-1} \tag{7}$$

since a unique S^{-1} always exists. Rather than attempt to solve for N_m—which requires a pseudo-inverse since mm^T is always singular—simply note that

$$N_m(m, S)m = E[f(x)] \tag{8}$$

This result is all that will be required to solve the problem at hand. To evaluate the expectations in Eqs. (7) and (8) requires an assumption about the probability density function of $x(t)$. Most often a gaussian density is assumed, although this need not be the case. Since the elements of x will in general be correlated, the assumption of a nongaussian joint density function is intractable for practical purposes. In addition, because of the filtering effect of the system integrations, the gaussian assumption tends to be the most widely applicable.

Replacing the nonlinear function of Fig. 1 by the describing function approximation given in Fig. 3, the differential equations of the resulting quasi-linear system for the propagation of the mean and covariance are

$$\dot{m} = N_m(m, S)m + b \tag{9}$$

and

$$\dot{S} = N_r(m, S)S + SN_r^T(m, S) + Q \tag{10}$$

where

$$E[u(t)u^T(\tau)] = Q(t)\delta(t-\tau)$$

given that $u(t)$ is the zero-mean white noise component of w and

$$b = E[w]$$

These equations are nonlinear and coupled through the mean and random signal matrix describing functions, N_m and N_r, respectively. Initialization is accomplished by associating the bias portion of $x(0)$ with $m(0)$ and the random portion with $S(0)$, where

$$S(0) = E[r(0)r^T(0)]$$

Equations (7–10) are the key equations of CADET, and the associated quasi-linear system model is illustrated in Fig. 4.

A few special relationships are worth noting. When the system is *linear*, $f(x) = Fx$ and Eqs. (7) and (8) immediately lead to the result $N_m = N_r = F$. Corresponding to Eqs. (9) and (10) are then the familiar equations for mean and covariance propagation in linear systems. In the case of scalar nonlinearities, N_m and N_r reduce to the bias and random-input describing functions defined elsewhere.[1] When the nonlinearity is *odd* [i.e., $f(-x) = -f(x)$], the effective gain to a small mean m in the presence of a multidimensional gaussian process, r, can be shown to be the same as the effective gain of that nonlinearity to r itself, that is,

$$\lim_{m\to 0} N_m = N_r(m=0) \tag{11}$$

Dynamical systems tend to contain more linear than nonlinear elements. This is a fortuitous occurrence, since low-pass linear filtering is necessary to insure that nongaussian nonlinearity outputs result in nearly gaussian nonlinearity inputs, as signals circulate in the system of interest. This so-called "filter hypothesis" is common to all describing function analyses.[1] Under the gaussian assumption, N_r can be computed[2] from the relationship

$$N_r(m,S) = (d/dm)E[f(x)] \tag{12}$$

This is indeed a useful relationship, since $E[f(x)]$ is also required in Eq. (8). It is in practice much easier to employ Eq. (12) than to solve Eq. (7) for $N_r(m,S)$. Even more important is the fact that Eq. (12) allows N_r to be formed by first replacing the individual nonlinear elements of $f(x)$ by the appropriate scalar describing function gains, and then forming the resulting quasi-linearized system F matrix. [Note from Eq. (12) that the F matrix associated with the linear portion of $f(x)$ will appear directly in N_r.] This is an extremely powerful property since a large number of describing functions have been catalogued,[1] which in many cases allows N_m and N_r to be formed directly from inspection of the system equations or block diagram.

The accuracy of the statistical linearization technique depends on the assumed nonlinearity input probability density function. Although it is not a requirement of CADET, the assumption of a gaussian process is a logical one. Thus, any nonlinear effect which produces a distribution having extreme skewness, or any other property in direct conflict with the bias plus gaussian process assumption, will degrade CADET performance. The only present method to test the accuracy of CADET is by comparison with Monte Carlo derived statistics. However, once the particular CADET model has been verified for one or a limited number of conditions, it can then be used with some confidence to study a wide range of system characteristics without resorting to additional Monte Carlo trials.

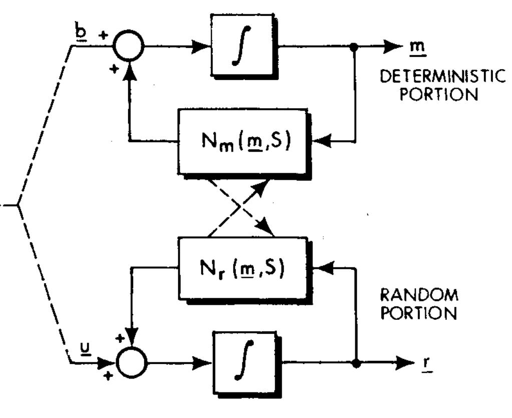

Fig. 4 Quasi-linear system model.

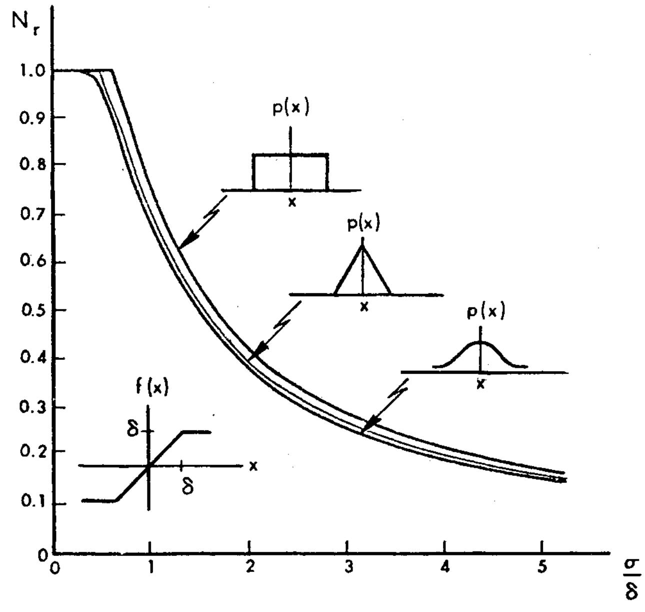

Fig. 5 Describing function gain sensitivity example.

It is particularly instructive to investigate the describing function gain sensitivity to various assumed input probability density functions. If the gain is relatively insensitive to the shape of the probability density function, it may be reasonable to also expect CADET performance sensitivity to be small. The zero-mean random-input describing functions for a limiter, computed for the indicated densities in Fig. 5, demonstrate a general insensitivity relative to the gaussian case. Similar results hold in other cases as well—a very encouraging behavior.

It is also quite easy and useful to show that if the probability density function $p(x)$ is comprised of elemental probability density functions $p_1(x)$ and $p_2(x)$, the resulting bias and random-input describing functions obtained with $p(x)$ are bounded by those obtained using $p_1(x)$ and $p_2(x)$ individually. Assuming that

$$p(x) = ap_1(x) + (1-a)p_2(x)$$

where $0 \leq a \leq 1$, the scalar bias and random-input describing functions are given by

$$N_m(b,\sigma) = aN_{m1}(b,\sigma_1) + (1-a)N_{m2}(b,\sigma_2) \tag{13}$$

$$N_r(b,\sigma) = \alpha N_{r1}(b,\sigma_1) + (1-\alpha)N_{r2}(b,\sigma_2) \tag{14}$$

where $0 \leq \alpha \leq 1$ and where N_{m1}, N_{r1}, N_{m2}, and N_{r2} are the mean and random-input describing functions associated with $p_1(x)$ and $p_2(x)$, respectively. The resulting describing functions are a simple linear combination of the describing functions obtained with the elemental density functions. This result suggests a method for computing the describing functions for a class of complex probability density functions.

Computation of Quasi-Linearized System Statistics

The computations of the mean and covariance of the quasi-linearized system are accomplished with Eqs. (9) and (10), which are a coupled set of time varying, nonlinear differential

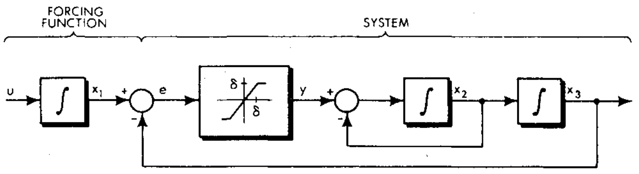

Fig. 6 Second-order system block diagram with error signal saturation.

equations. If all system nonlinearities are odd and all initial and input mean values are zero, there is no need to propagate the mean value equation since all mean values will be zero. However, in the case of nonzero mean inputs or asymmetric nonlinearities for which rectification can produce nonzero mean state values, the implementation of Eq. (9) is required.

Considerable attention has been given in the literature to methods for solving matrix covariance equations.[5,6] However, many of the techniques proposed consider the time-invariant system for which the transition matrix approach or some type of series solution is well suited. For the solution of time-varying nonlinear differential equations, the most straightforward and in many cases the best method is direct numerical integration. Self-starting rather than predictor-corrector methods are often preferred, particularly if the system contains discrete elements which require frequent updates. Of the self-starting methods, fourth-order Runge-Kutta tends to be the most popular. However, accuracy and stability of the solution is dependent upon the integration step size, initial conditions, system dynamics, and the process-noise covariance matrix. In many cases, the required accuracy can be achieved simply through the use of double-precision computer words and reduced integration intervals.

Predictor-corrector methods or automatic integration interval computation can be used to eliminate the requirement to specify the integration interval. However, the error criterion to be used in computing the integration interval will have to be specified. In addition, most popular predictor-corrector methods —e.g., Milne, Hamming, etc.—are not self-starting and generally use Runge-Kutta methods to generate the required starting values. Since there are no prescribed rules for the selection of the "best" numerical integration method for a particular problem, one must often resort to experimentation with various techniques.

If the describing function matrices N_m and N_r are assumed to be piecewise-constant over "small" time intervals, linear time-invariant system techniques can be used to approximate the solution. These techniques tend to be computationally superior to direct integration when applied to linear time-invariant systems. However, this is not necessarily true in the case of CADET and should be tested against direct integration methods.

The procedure for the mechanization of CADET is straightforward once the describing functions have been obtained. In fact, for many systems which contain nonlinear elements of the type already catalogued,[1] the describing function matrices are simply obtained by substituting the appropriate scalar describing functions for the nonlinearities and forming the system matrix. By including the required numerical calculations for the describing function gains in computer subroutines or using table look-up methods, Eqs. (9) and (10) can be solved using any of the previously mentioned methods. A direct and efficient method for the analytic calculations of all piecewise-linear and modified polynomial nonlinearities, which represents a very significant family of nonlinear functions, is contained elsewhere.[1] The required operations are well suited to digital computer solution.

Illustrative Examples

As a demonstration of the utility of CADET, two relatively simple but practical systems are considered. CADET is used to evaluate the rms system response of each example, which is then compared with the results of a 200-run Monte Carlo direct simulation.

Second-Order System with Error Signal Saturation

A descriptive block diagram of the system is given in Fig. 6. The system forcing function is a zero-mean gaussian random walk process having a mean-squared value which is proportional to time, and the modeled nonlinearity is error signal saturation.

The quasi-linearized system can be represented in state vector notation as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ N(\sigma_e) & -1 & -N(\sigma_e) \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

where the gaussian random-input describing function, $N(\sigma_e)$, is given in Fig. 5.

Using the system as defined in Eq. (15), the CADET simulation was run for a problem duration of 50 sec with a 0.1-sec integration interval. The linear ($\delta = \infty$) and highly nonlinear ($\delta = 0.1$) system rms outputs (x_3) are presented as a function of time in Fig. 7 in addition to the results of a 200-run Monte Carlo simulation. The relative error between the CADET and Monte Carlo methods is less than 2.5%.

Terminal Homing Missile with Airframe Acceleration Saturation

This example is representative of a terminal homing missile performance evaluation problem which historically required the Monte Carlo method. A constant velocity interceptor missile utilizing proportional navigation is assumed. A simple first-order lag is used to model the guidance system and autopilot dynamics. (The inclusion of higher-order dynamics would not compromise the application of CADET.) Lateral acceleration saturation of the missile airframe is the modeled nonlinearity and is in practice one of the primary sources of miss distance if the target is performing evasive maneuvers. The target is assumed to have a constant velocity and a random lateral acceleration maneuver which is characterized by a first-order

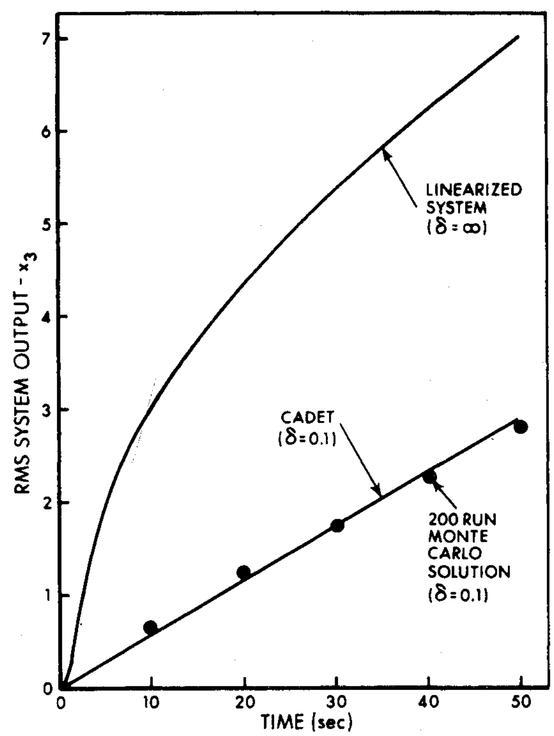

Fig. 7 Simulation results for the linear, quasi-linear and Monte Carlo solutions.

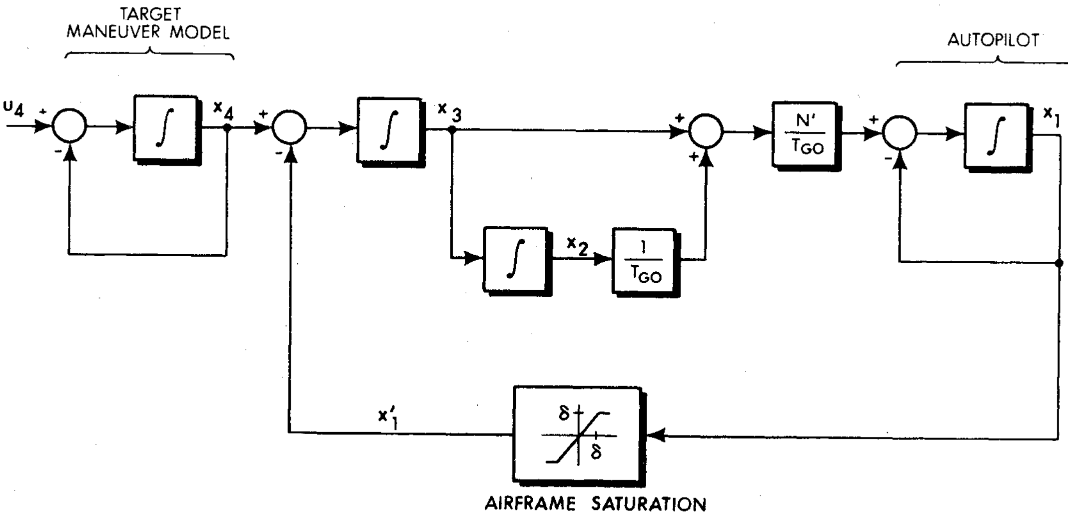

Fig. 8 Kinematic guidance loop with airframe acceleration saturation—x_1 = commanded airframe acceleration (ft/sec^2); x_1' = actual airframe acceleration (ft/sec^2); x_2 = relative lateral separation (ft); x_3 = relative separation rate (fps); x_4 = target lateral acceleration (ft/sec^2); u_4 = white noise [(ft/sec^2)^2/Hz].

Markov process with a standard deviation of 5 g. The fundamental figure-of-merit for missile performance is miss distance, which is defined as the relative missile-to-target separation at the point of closest approach. The rms miss distance is determined as a function of the missile acceleration saturation level with the CADET simulation and compared with Monte Carlo results. For simplicity, the "head-on" intercept case is considered. A mathematical block diagram of the system under consideration is given in Fig. 8 where T_{GO} is the time-to-intercept and N' is the guidance ratio ($N' = 3$ in this example).

The CADET and Monte Carlo simulation results for the target acceleration, x_4, and the relative separation, x_2, are presented in Fig. 9 for the linearized system ($\delta = \infty$) along with a 10-g and 1-g airframe saturation level. The same computer generated noise sequence for the 200 Monte Carlo runs was used for each of the airframe saturation levels. Therefore,

Fig. 9 Simulation results for various levels of airframe acceleration saturation.

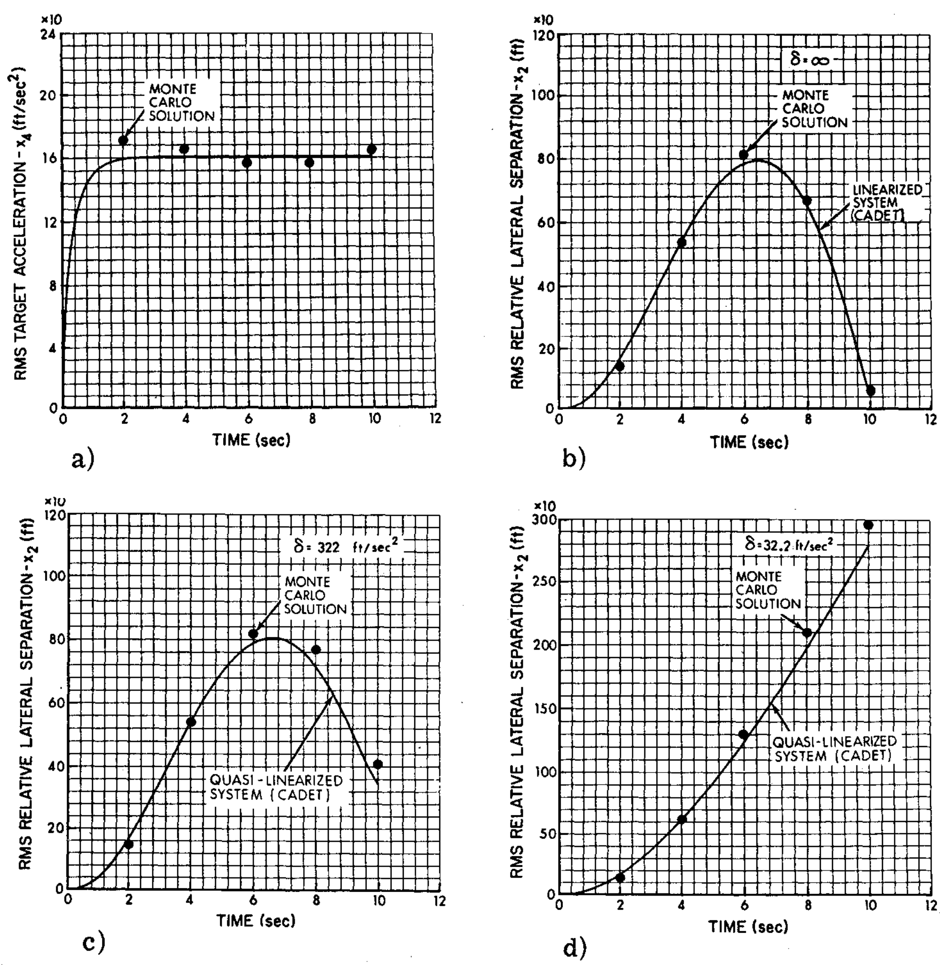

Fig. 9a is common to each case. The rms miss distance is given by the relative separation at 10 sec ($T_{GO} = 0$).

The results clearly demonstrate the applicability of CADET to the evaluation of terminal miss distance for an acceleration-limited missile. The CADET and Monte Carlo results agree to within 10%. In addition, it is interesting to note that the miss distance obtained with the linearized system model (Fig. 9b) is entirely misleading in the presence of missile lateral acceleration limitations. The fundamental reason for the breakdown of the linear analysis is the $T_{GO}{}^{-1}$ dependency of the loop gain which causes a rapid increase in the commanded airframe acceleration near intercept. Guidance control is effectively lost when the airframe saturates, and the result is a marked increase in miss distance as compared to that obtained by use of linear system models.

Conclusions

CADET is a powerful new approach to the direct statistical analysis of nonlinear systems. It is applicable to high-order systems with multiple nonlinearities, multiple inputs, and non-gaussian statistics. Its advantages are simplicity and economy; a disadvantage is that it is not exact. Another disadvantage, common to all describing function approaches, is that no suitable accuracy analysis exists. It follows that some Monte Carlo trials should always be made to validate the performance of a CADET simulation. On balance, however, the advantages of CADET are thought to be sufficient to ensure its substantial application to the direct statistical analysis of nonlinear systems.

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Minimax Design of Kalman-Like Filters in the Presence of Large-Parameter Uncertainties

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The Kalman filter has been used in many applications; however, practical implementation of the filter has required exact knowledge of the various system parameters (input and measurement noise covariance) in order to yield optimum performance. This paper develops a minimax technique for the direct synthesis of Kalman-like estimators when there are large uncertainties in the a priori statistics of the plant and measurement noises. Both continuous and discrete estimators are considered. General properties of the filters that satisfy the various minimax performance indices are discussed and a number of examples of both continuous and discrete applications are then presented to demonstrate the technique.

Introduction

THE Kalman filter,[1,2] designed to estimate the states of a linear system driven by white noise using measurements corrupted by white noise, has been used in many applications. However, practical implementation of the filter has required specific knowledge of the various system statistical parameters (input and measurement noise covariance) in order to yield optimum performance. If there is significant uncertainty associated with these parameters, a sensitivity analysis is usually carried out to evaluate the actual filter performance. It is the intent of the work presented herein to guide the designer through a systematic procedure which will yield a filter that performs in an acceptable fashion when there is significant uncertainty in the various system parameters. Very general restrictions are placed upon these uncertainties, and one will find that they are applicable in most practical situations.

The basis for the results in this paper is contained in several papers by D'Appolito and Hutchinson,[3-6] where the idea of

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